AEM JOURNAL.ORG

Some integrals involving squares of Bessel functions and generalized Legendre polynomials

E. Matagne[1]
[1]UCLouvain/EPL/iMMC/MEED, Place du Levant 2 bte L5.04.02, B-1348 Louvain-la-Neuve, Belgium
Corresponding author: E. Matagne (e-mail: ernest.matagne@uclouvain.be)

ABSTRACT This paper develops new integral formulas intended for detailed studies of electromagnetic normal modes in spherical and spherical annular cavities.

INDEX TERMS Bessel functions, Generalized Legendre polynomials, Integral, Square.

I. INTRODUCTION

TO study electromagnetic normal modes in spherical or spherical annular cavities, we have developed formulas that do not seem to be listed in the classical literature. As these formulas could be applied in several scientific fields, we think it is useful to publish them independently of their application. These formulas allow to find expressions for the energy, thrust and losses of all electromagnetic normal modes in spherical or annular spherical cavities. Computation of energy and losses is important in the study of electromagnetic resonators, whereas their expressions have to our knowledge only been developed for the first few modes in spherical cavities [1]. Such cavities are now in development for use in THz devices (see e.g. [2]). In theoretical physics, the availability of expressions for the thrust applied by each mode on a spherical surface should allow a more direct calculation of the Casimir effect than the classical calculation based only on the energy variation [3]. Casimir effect is now studied in different structures with spherical symmetry (see e.g. [4]). It has also practical implications in NEMS (nano electro mechanical systems), sensors and material sciences (see e.g. [5]).

II. QUADRATIC INTEGRALS WITH RESPECT TO BESSEL FUNCTIONS

Bessel functions of order ν are solution of the differential equation [6](9.1.1)

$$x^2\Psi"_\nu(x) + x\Psi'_\nu(x) + (x^2 - \nu^2)\Psi_\nu(x) = 0 \qquad (1)$$

where $'$ denotes the first derivative and $"$ the second derivative. For our purposes, we can always consider that x is a positive real variable.

$$x > 0 \qquad (2)$$

The general solution of (1) is a linear combination of Bessel functions of first and second kind, i.e.

$$\Psi_\nu(x) = AJ_\nu(x) + BY_\nu(x) \qquad (3)$$

where A and B are two real parameters with any values.

A. GENERALIZATION OF THE LOMMEL INTEGRALS

We are looking for an analytical solution of the integral

$$\int x\Psi_\nu^2(\alpha x)dx \qquad (4)$$

In the literature, what are called Lommel integrals are similar to (4) but are restricted to the special case of the Bessel function of the first kind $J_\nu(x)$ and often limited to a finite integration interval $[0, a]$. In fact, the two integrals

$$\int_0^a xJ_\nu(\alpha x)J_\nu(\beta x)dx \qquad (5)$$

and

$$\int_0^a xJ_\nu^2(\alpha x)dx \qquad (6)$$

are named Lommel integrals [7]. The methods for solving the Lommel integrals (5) (6), as detailed in [8] or [9], can however be generalized without difficulty to the calculation of (4).

One easily obtains

$$\int x\Psi_\nu^2(\alpha x)dx = \frac{x^2}{2}[\Psi_\nu'^2(\alpha x) + (1-\frac{\nu^2}{\alpha^2x^2})\Psi_\nu^2(\alpha x)] + cst \qquad (7)$$

Expression (7) generalizes the Lommel integral (6) as quoted in [1]. To obtain the generalization of that expression as quoted in [7], we still have to replace the term $\Psi_\nu'^2(\alpha x)$ so as not to keep any derivative function. To do this, we use the last two of the recurrence formulas given in [6](9.1.27), namely:

$$\Psi_\nu'(x) = \Psi_{\nu-1}(x) - \nu x^{-1}\Psi_\nu(x) \tag{8}$$

and

$$\Psi_\nu'(x) = -\Psi_{\nu+1}(x) + \nu x^{-1}\Psi_\nu(x) \tag{9}$$

The product of (8) and (9) yields

$$\Psi_\nu'^{\,2}(x) = [\Psi_{\nu-1}(x) - \nu x^{-1}\Psi_\nu(x)][-\Psi_{\nu+1}(x) + \nu x^{-1}\Psi_\nu(x)] \tag{10}$$

or

$$\Psi_\nu'^{\,2}(x) = -\Psi_{\nu-1}(x)\Psi_{\nu+1}(x) - \nu^2 x^{-2}\Psi_\nu^2(x) + \nu x^{-1}\Psi_\nu(x)[\Psi_{\nu-1}(x) + \Psi_{\nu+1}(x)] \tag{11}$$

Furthermore, subtracting (9) from (8), we obtain:

$$\Psi_{\nu-1}(x) + \Psi_{\nu+1}(x) = 2\nu x^{-1}\Psi_\nu(x) \tag{12}$$

Inserting (12) into (11), we obtain:

$$\Psi_\nu'^{\,2}(x) = -\Psi_{\nu-1}(x)\Psi_{\nu+1}(x) + \nu^2 x^{-2}\Psi_\nu^2(x) \tag{13}$$

It remains to introduce (13) into (7) to obtain

$$\int x\Psi_\nu^2(\alpha x)dx = \frac{x^2}{2}[\Psi_\nu^2(\alpha x) - \Psi_{\nu-1}(\alpha x)\Psi_{\nu+1}(\alpha x)] + cst \tag{14}$$

which generalizes the Lommel integral as quoted in [7]. Equations (7) and (14) generalize equations (6.52) and (6.53) of reference [9].

B. OTHER QUADRATIC INTEGRAL

We are now looking for an analytical solution of the integral

$$\int\{\frac{\nu^2 - 1/4}{r}\Psi_\nu^2(kr) + [\frac{d}{dr}(\sqrt{r}\Psi_\nu(kr))]^2\}dr \tag{15}$$

We may write

$$\frac{d}{dr}(\sqrt{r}\Psi_\nu(kr)) = \frac{1}{2\sqrt{r}}\Psi_\nu(kr)) + \sqrt{r}k\Psi_\nu'(kr)) \tag{16}$$

so

$$[\frac{d}{dr}(\sqrt{r}\Psi_\nu(kr))]^2 = \frac{1}{4r}\Psi_\nu^2(kr) + k\Psi_\nu(kr)\Psi_\nu'(kr) + rk^2\Psi_\nu'^{\,2}(kr) \tag{17}$$

By substituting (17) in (15), we get

$$\int ...dr = \int\{\frac{\nu^2}{r}\Psi_\nu^2(kr) + k\Psi_\nu(kr)\Psi_\nu'(kr) + rk^2\Psi_\nu'^{\,2}(kr)\}dr \tag{18}$$

By grouping the first and third terms of the integrand, we get

$$\int ...dr = \int\{\frac{\nu^2}{r}\Psi_\nu^2(kr) + rk^2\Psi_\nu'^{\,2}(kr)\}dr + \int\frac{d}{dr}\frac{1}{2}\Psi_\nu^2(kr)dr \tag{19}$$

so

$$\int ...dr = \int\{\frac{\nu^2}{r}\Psi_\nu^2(kr) + rk^2\Psi_\nu'^{\,2}(kr)\}dr + \frac{1}{2}\Psi_\nu^2(kr) \tag{20}$$

Moreover, by squaring the first two recurrence relations given in [6](9.1.27) and summing them member to member, we obtain after division by 4 and substitution of z by kr

$$\Psi_\nu'^{\,2}(kr) + \frac{\nu^2}{k^2r^2}\Psi_\nu^2(kr) = \frac{1}{2}[\Psi_{\nu-1}^2(kr) + \Psi_{\nu+1}^2(kr)] \tag{21}$$

Multiplying (21) by $k^2 r$ and substituting the result in (20), we get

$$\int ...dr = \frac{k^2}{2}\int r\Psi_{\nu-1}^2(kr)dr + \frac{k^2}{2}\int r\Psi_{\nu+1}^2(kr)dr + \frac{1}{2}\Psi_\nu^2(kr) \tag{22}$$

We recognize in (22) two generalized Lommel integrals (14), therefore

$$\begin{aligned}\int ...dr &= \frac{k^2}{2}\frac{r^2}{2}[\Psi_{\nu-1}^2(kr) - \Psi_{\nu-2}(kr)\Psi_\nu(kr)] \\ &+ \frac{k^2}{2}\frac{r^2}{2}[\Psi_{\nu+1}^2(kr) - \Psi_\nu(kr)\Psi_{\nu+2}(kr)] + \frac{1}{2}\Psi_\nu^2(kr) + cst \\ &= \frac{k^2r^2}{2}\frac{1}{2}[\Psi_{\nu-1}^2(kr) + \Psi_{\nu+1}^2(kr)] \\ &- \frac{k^2r^2}{4}\Psi_\nu(kr)[\Psi_{\nu-2}(kr) + \Psi_{\nu+2}(kr)] + \frac{1}{2}\Psi_\nu^2(kr) + cst\end{aligned} \tag{23}$$

By using relation (21) in the opposite direction, one obtains

$$\int ...dr = \frac{\nu^2}{2}\Psi_\nu^2(kr) + \frac{k^2r^2}{2}\Psi_\nu'^{\,2}(kr) - \frac{k^2r^2}{4}\Psi_\nu(kr)[\Psi_{\nu-2}(kr) + \Psi_{\nu+2}(kr)] + \frac{1}{2}\Psi_\nu^2(kr) + cst \tag{24}$$

Now, by substituting x by kr in the equation (47) given in the appendix, and by introducing the result in (24), we obtain

$$\begin{aligned}\int ...dr &= \frac{\nu^2+1}{2}\Psi_\nu^2(kr) + \frac{k^2r^2}{2}\Psi_\nu'^{\,2}(kr) \\ &- \frac{k^2r^2}{4}\Psi_\nu(kr)[-\frac{4}{kr}\Psi_\nu'(kr) + (\frac{4\nu^2}{k^2r^2} - 2)\Psi_\nu(kr)] + cst \\ &= \frac{\nu^2+1}{2}\Psi_\nu^2(kr) + \frac{k^2r^2}{2}\Psi_\nu'^{\,2}(kr) \\ &+ kr\Psi_\nu(kr)\Psi_\nu'(kr) + [-\nu^2 + \frac{k^2r^2}{2}]\Psi_\nu^2(kr) + cst\end{aligned} \tag{25}$$

or

$$\int ...dr = [2kr\Psi_\nu'(kr) + \Psi_\nu(kr)][\frac{kr}{4}\Psi_\nu'(kr) + \frac{3}{8}\Psi_\nu(kr)] + [\frac{1}{8} - \frac{(\nu)^2}{2} + \frac{k^2r^2}{2}]\Psi_\nu^2(kr) + cst \tag{26}$$

and finally

$$\int\{\frac{\nu^2-1/4}{r}\Psi_\nu^2(kr)+[\frac{d}{dr}(\sqrt{r}\Psi_\nu(kr))]^2\}dr$$
$$=[2kr\Psi'_\nu(kr)+\Psi_\nu(kr)][\frac{kr}{4}\Psi'_\nu(kr)+\frac{3}{8}\Psi_\nu(kr)]$$
$$+[\frac{k^2r^2}{2}-\frac{\nu^2-1/4}{2}]\Psi_\nu^2(kr)+cst \quad (27)$$

The writing of (27) is well suited to easily take into account the boundary conditions encountered in electromagnetism for spherical or annular spherical cavities.

III. QUADRATIC INTEGRALS WITH RESPECT TO GENERALIZED LEGENDRE POLYNOMIALS

For the calculation of the energy and forces of the spherically symmetric electromagnetic normal modes, we also had to solve integrals related to the generalized Legendre polynomials $P_\ell^m(x)$. We use the definition of these polynomials given in [6]. For our purpose, it was sufficient to consider the values:

$$\ell=1....\infty \quad (28)$$

$$m=0....\ell \quad (29)$$

A. FIRST CASE

The first integral we consider is

$$\int_0^\pi \sin\theta[\frac{d}{d\theta}P_\ell^m(\cos\theta)]^2d\theta=\int_0^\pi \sin^3\theta[P'^m_\ell(\cos\theta)]^2d\theta \quad (30)$$

or, by performing the change of variable $u=\cos\theta$,

$$\int_0^\pi \sin\theta[\frac{d}{d\theta}P_\ell^m(\cos\theta)]^2d\theta=-\int_1^{-1}(1-u^2)[P'^m_\ell(u)]^2du$$
$$=\int_{-1}^{1}(1-u^2)[P'^m_\ell(u)]^2du \quad (31)$$

Using the formula (14.10.4) given in [10]:

$$(1-u^2)P'^m_\ell(u)=(m-\ell-1)P'^m_\ell(u)+(\ell+1)P_\ell^m(u) \quad (32)$$

we can write (31) as

$$\int_0^\pi \sin\theta[\frac{d}{d\theta}P_\ell^m(\cos\theta)]^2d\theta$$
$$=\int_{-1}^{1}(m-\ell-1)P_{\ell+1}^m(u)P'^m_\ell(u)du$$
$$+\int_{-1}^{1}(\ell+1)P_\ell^m(u)P'^m_\ell(u)du \quad (33)$$

Since each term contains a product of functions with the same upper index ℓ we can use the formulas given in [11] without worrying about the difference in definition with [6]. So, we can use formulas (35) and (37) of [11] to solve the integrals of the right-hand side of (33). We obtain

$$\int_0^\pi \sin\theta[\frac{d}{d\theta}P_\ell^m(\cos\theta)]^2d\theta$$
$$=(m-\ell-1)\left\{\delta_{0,m}-\frac{(\ell+m)!}{(\ell-m)!}\right\}$$
$$+(\ell+1)\left\{\delta_{0,m}-\frac{(\ell+m)!}{(2\ell+1)(\ell-m)!}\right\} \quad (34)$$

where $\delta_{0,m}=0$, unless $m=0$ in which case $\delta_{0,m}=1$. If $m=0$, then we have

$$\int_0^\pi \sin\theta[\frac{d}{d\theta}P_\ell^m(\cos\theta)]^2d\theta$$
$$=(\ell+1)\left\{1-\frac{1}{(2\ell+1)}\right\}=\frac{\ell(\ell+1)}{\ell+1/2} \quad (35)$$

If $m\neq 0$, then we have

$$\int_0^\pi \sin\theta[\frac{d}{d\theta}P_\ell^m(\cos\theta)]^2d\theta$$
$$=(\ell-m+1)\frac{(\ell+m)!}{(\ell-m)!}-(\ell+1)\frac{(\ell+m)!}{(2\ell+1)(\ell-m)!} \quad (36)$$

so

$$\int_0^\pi \sin\theta[\frac{d}{d\theta}P_\ell^m(\cos\theta)]^2d\theta=[\frac{\ell(\ell+1)}{(\ell+1/2)}-m]\frac{(\ell+m)!}{(\ell-m)!} \quad (37)$$

Formula (37) reduces to (35) when $m=0$, so that (37) is valid without restriction on m.

B. SECOND CASE

If $m\neq 0$, using formula (8.14.14) given in [6], we obtain

$$\int_0^\pi \frac{1}{\sin\theta}[P_\ell^m(\cos\theta)]^2d\theta=\frac{(\ell+m)!}{m(\ell-m)!} \quad (38)$$

From (38), it is easy to derive a formula that hold for all values of m given by (29):

$$\int_0^\pi \frac{m^2}{\sin\theta}[P_\ell^m(\cos\theta)]^2d\theta=m\frac{(\ell+m)!}{(\ell-m)!} \quad (39)$$

IV. CONCLUSIONS

At the end of this study, integrals given in (7) or (14),(27), (37) and (39), together with (8.14.13) given in [6], are all the integral formulas needed to obtain analytical expressions for the energy, thrust and losses of all electromagnetic normal modes in spherical or annular spherical structures. As stated in the introduction, the results given in this paper can be useful in different areas of theoretical physics and applied physics.

APPENDIX: DOUBLE RECURRENCES ON BESSEL FUNCTIONS

The last two equations found in [6] (9.1.27) can be written

$$\Psi_{\nu\pm1}(x)=\mp\Psi'_\nu(x)+\frac{\nu}{x}\Psi_\nu(x) \quad (40)$$

By deriving these relations, we obtain

$$\Psi'_{\nu\pm1}(x) = \mp\Psi"_{\nu}(x) + \frac{\nu}{x}\Psi'_{\nu}(x) - \frac{\nu}{x^2}\Psi_{\nu}(x) \quad (41)$$

Eliminating the second derivatives by the equation of definition of the Bessel functions (1), one has

$$\Psi'_{\nu\pm1}(x) = \pm\frac{1}{x}\Psi'_{\nu}(x) \pm (1 - \frac{\nu^2}{x^2})\Psi_{\nu}(x) + \frac{\nu}{x}\Psi'_{\nu}(x) - \frac{\nu}{x^2}\Psi_{\nu}(x) \quad (42)$$

or

$$\Psi'_{\nu\pm1}(x) = \frac{\nu\pm1}{x}\Psi'_{\nu}(x) \mp (\frac{\nu(\nu\pm1)}{x^2} - 1)\Psi_{\nu}(x) \quad (43)$$

Let us go back to the equations (40). By replacing ν by $\nu\pm1$, we obtain

$$\Psi_{\nu\pm2}(x) = \mp\Psi'_{\nu\pm1}(x) + \frac{\nu\pm1}{x}\Psi_{\nu\pm1}(x) \quad (44)$$

Introducing (40) and (43) in (44), we obtain

$$\Psi_{\nu\pm2}(x) = \mp\frac{\nu\pm1}{x}\Psi'_{\nu}(x) + (\frac{\nu(\nu\pm1)}{x^2} - 1)\Psi_{\nu}(x) \mp\frac{\nu\pm1}{x}\Psi'_{\nu}(x) + \frac{\nu(\nu\pm1)}{x^2}\Psi_{\nu}(x) \quad (45)$$

or, finally

$$\Psi_{\nu\pm2}(x) = \mp2\frac{\nu\pm1}{x}\Psi'_{\nu}(x) + (2\frac{\nu(\nu\pm1)}{x^2} - 1)\Psi_{\nu}(x) \quad (46)$$

We observe remarkable equality

$$\Psi_{\nu+2}(x) + \Psi_{\nu-2}(x) = -\frac{4}{x}\Psi'_{\nu}(x) + (\frac{4\nu^2}{x^2} - 2)\Psi_{\nu}(x) \quad (47)$$

Acknowledgements : Thanks to Alexandre Matagne and Pr. Francis Labrique for their help concerning the use of English language.

REFERENCES

[1] R.A. Yadav and I.D. Singh, "Normal modes and quality factors of spherical dielectric resonators: I - Shielded dielectric sphere," PRAMANA journal of physics, vol. 62, no. 6, pp. 1255-1271, June 2004, 10.1007/BF02704438.

[2] D.W. Vogt and R. Leonhardt, "Ultra-high Q terahertz whispering-gallery modes in a silicon resonator," APL Photonics, vol. 3, 051702, 2018, https://doi.org/10.1063/1.5010364

[3] T. H. Boyer, "Quantum Electromagnetic Zero-Point Energy of a Conducting Spherical Shell and the Casimir Model for a Charged Particle," Physical Review, vol. 174, no. 5, pp. 1764-1776, October 1963, 10.1103/PhysRev.174.1764.

[4] Y. Avni and U. Leonhardt, "Casimir self-stress in dielectric sphere," Annals of Physics, vol. 395, pp. 326-340, 2018, https://doi.org/10.1016/j.aop.2018.06.002

[5] T. Gong, M.R. Corrado, A.R. Mahbub, C. Shelden and J.N. Munday "Recent progress in engineering the Casimir effect - application to nanophotonics, nanomechanics and chemistry," Nanophotonics, vol. 10, no. 1, pp. 523-536, 2021, https://doi.org/10.1515/nanoph-2020-0425

[6] M. Abramowitz and J. A. Stegun, Handbook of Mathematical Functions, Dover Publications, INC, NY, 1965.

[7] WolframMathWorld, Lommel's Integrals, [Online]. Available: http://mathworld.wolfram.com/LommelsIntegrals.html, Accessed on: May 8 2021

[8] V. G. Bagrov, V. V. Belov, V. N. Zadorozhnyi and A. Yu. Trifonov, "Methods of mathematical physics. Special Functions. Equations of Mathematical Physics.,"Tomsk Polytechnic University Publishing House, 2012.

[9] F. Bowman, Introduction to Bessel Functions, Dover Publications, INC, NY, 1958.

[10] T. M. Dunster, "Chapter 14 Legendre and Related Functions," in NIST Digital Library of Mathematical Functions, NIST Digital Library of Mathematical Functions [Online]. Available: http://dlmf.nist.gov/14.10, Accessed on May 8 2021

[11] S. N. Samaddar, "Some Integrals Involving Associated Legendre Functions," Mathematics of Computation, vol. 28, no. 125, pp. 257-263, January 1974.